Mathematical Formulation of the Maximum Ordinality Principle

As already anticipated, the Maximum Ordinality Principle asserts that:

"Every System tends to Maximize its own Ordinality, including that of the surrounding habitat".

Such a verbal enunciation can correspondently be expressed in formal terms by means of three intimately correlated basic equations:

i) The first equation, which formally expresses the general tendency toward the Maximum Ordinality, can be formulated as follows:

$$(\tilde{d}/\tilde{d}t)^{(\tilde{m}/\tilde{n})} \{\tilde{r}\}_{s} = 0 \qquad (\tilde{m}/\tilde{n}) \to Max$$
 (1)

where: $(\tilde{d}/\tilde{d}t)$ is the symbol of the "incipient" derivative; (\tilde{m}/\tilde{n}) is the Ordinality of the System, which represents the Structural Organization of the same in terms of Co-Productions, Inter-Actions, Feed-Backs; while $\{\tilde{r}\}_s$ is the proper Space of the System (see also Eq. (3)).

The symbol $(m/n) \to Max$ in Eq. (1) has to be understood as follows: when a Self-organizing System, persistently propending toward the Maximum Ordinality conditions, effectively reaches such very special conditions, it presents itself as being self-structured in a radically different way with respect to its initial Ordinality. This is because the latter has evolved according to the following Trans-formation

$$(\widetilde{n}/\widetilde{n}) \to \{\{\widetilde{2}/\widetilde{2}\} \uparrow \{\widetilde{2}\uparrow\}\} \uparrow \widetilde{N}$$
 (1'),

where: $\{\tilde{2}/\tilde{2}\}$ represents a "binary-duet" coupling; the Ordinal power $\{\tilde{2}/\tilde{2}\}$ indicates the "perfect specularity" of the previous "binary-duet" structure; while \hat{N} indicates the Ordinal Over-structure of the \hat{N} elements of the System considered as a Whole (this is the reason for the "tilde" notation);

ii) The second equation, which is intimately related to the previous one, expresses an internal Harmony Condition to the System (or, alternatively, the internal Ordinal "Stability" of the System), for each level of Ordinality achieved.

For the sake of simplicity and clarity the latter is formulated with reference to any single couple of elements, when structured in a "binary-duet" relationship:

$$(\tilde{d}/\tilde{d}t)^{(\tilde{2}/\tilde{2})}\{\{\tilde{r}\}\otimes(\tilde{d}/\tilde{d}t)^{(\tilde{2}/\tilde{2})}\{\tilde{r}\}\}=0$$
(2).

This equation asserts that the proper Space of the System (at present considered as being made up of two sole elements) is coupled with its specific Generativity in such a way as to originate a comprehensive Generative Capacity which is always in equilibrium.¹

This is also the equation which represents a fundamental condition for the intrinsic Ordinal Stability of the System as a Whole and also generates the afore-mentioned "perfect specularity" of the System (understood as internal Harmony Relationships). A perfect specularity which, in the case of two sole elements, is represented by the Ordinal structure $\{\{\tilde{2}/\tilde{2}\} \uparrow \{\tilde{2}\uparrow\}\}$, while in the case of \tilde{N} elements is represented by the right hand side of Eq. (1');

iii) finally, the third equation, which defines the fundamental Reference Space (of the System):

$$\widetilde{\{r\}} = \widetilde{\{x \otimes i \oplus y \otimes j \oplus z \otimes k\}}$$
(3),

where the coordinates (x, y, z) are understood as being the exit of a Generative Process (this is the reason for the tilde notation); the symbols \oplus and \otimes express more intimate relationships between the same coordinates: both in terms of sum (\oplus) and in terms of (relational) product (\otimes) with respect to the traditional versors \overrightarrow{i} , \overrightarrow{j} , \overrightarrow{k} ; the symbol "=", usually replaced by "=" only for the sake of simplicity, indicates an "assignation" condition. This is because the second hand of Eq. (3) is understood as *one sole entity*, thus representing "something more" than the "sum" of its component (this is also the reason for the brackets). Consequently, definition (3) represents, in reality, an Over-definition.

For practical purposes, however, it is more useful to adopt an equivalent representation, which is obtainable from a generalized version of Moivre's formula

$$\{\tilde{r}\} = \{\tilde{\rho} \otimes \tilde{i} \otimes e^{\tilde{\varphi} \otimes \tilde{j}} \otimes e^{\tilde{g} \otimes \tilde{k}}\}$$
 (3'),

where the coordinates $(\tilde{\rho}, \tilde{\varphi}, \tilde{\vartheta})$ are still considered as being the exit of a Generative Process, whereas the traditional versors \tilde{i} , \tilde{j} , \tilde{k} are now replaced by three unit spinors \tilde{i} , \tilde{j} , \tilde{k} (see mathematical Appendix). Representation (3') is very similar (albeit not strictly equivalent) to a system of three complex numbers, characterized by one real unit (\tilde{i}) and two imaginary units (\tilde{j}) and (\tilde{k}) . This can easily be recognized by simply considering the specific properties of these spinors given in Appendix.

Under these conditions, the solution to Eq. (1) (together with associated Eqs. (2) and (3')) can be expressed in the form of the following exponential Ordinal Matrix

$$\{\tilde{r}\}_{s} = e^{\begin{bmatrix} \tilde{\alpha}_{11}(t) & \tilde{\alpha}_{12}(t) & \dots & \tilde{\alpha}_{1N}(t) \\ \tilde{\alpha}_{21}(t) & \tilde{\alpha}_{22}(t) & \dots & \tilde{\alpha}_{2N}(t) \\ \dots & \dots & \dots & \dots \\ \tilde{\alpha}_{N1}(t) & \tilde{\alpha}_{N2}(t) & \dots & \tilde{\alpha}_{NN}(t) \end{bmatrix}}$$

 $^{^{1}}$ The symbol \otimes represents a more general form of "vector" product. However, at this stage of formulation, it can be considered as being perfectly equivalent to the traditional vector product.

in which any element α_{ij} is characterized by the Ordinality $\{\tilde{2}/\tilde{2}\} \uparrow \{\tilde{2}\uparrow\}$.

The procedure that leads to this solution is given in Appendix, where it is also shown how the search for such a solution is extremely facilitated not only by the structure of Eqs. (1) and (2), but also (and especially) by the conception of the basic reference space $\{r\}$ represented by Eq. (3').

The Ordinal Matrix (4) reflects the fact that the Relationships between the different parts of the System cannot be reduced to mere "functional" relationships between the corresponding cardinal quantities. This is because such quantities always "vehicle" something else, which leads us to term those relationships as "Ordinal" relationships. The term "Ordinal" thus explicitly reminds us that each part of the System is related to the others essentially because, prior to any other aspect, it is related to the Whole or, even better, it is "ordered" to the Whole. This is also the reason why the most important terms, when understood in such an Ordinal sense, are usually *capitalized* to expressly point out such a fundamental concept.

In such a perspective, each element of the Ordinal Matrix can be interpreted as being Inter-Acting (in Ordinal terms) with all the other elements of the System. In addition, the adoption of an *internal reference system* reveals that the afore-mentioned *perfect specularity* is a property which also characterizes the Ordinal Matrix as a

Whole. This is also the reason why the elements α_{ij} satisfy the following specularity conditions

$$\{\tilde{\alpha}_{ij}(t)\}^{\{\{\tilde{2}/\tilde{2}\}\uparrow\{\tilde{2}\uparrow\}\}} = \{\tilde{\alpha}_{ji}(t)\}^{\{\{\tilde{2}/\tilde{2}\}\uparrow\{\tilde{2}\uparrow\}\}}$$

$$(5),$$

which represent something more than the traditional symmetry $\alpha_{ij}(t) = \alpha_{ji}(t)$.

The adoption of an internal reference system, on the other hand, is an assumption which is strictly conform to the *Holistic* Approach (always subjacent to the Maximum Ordinality Principle). Such an assumption leads to recognize that $\alpha_{ii} = 0$ (for i =1, 2,...N), that is the main diagonal reduces to a sequence of zeros.

In addition, all the elements α_{ij} must satisfy the associated *Harmony Conditions*. These can be expressed

through some Correlating Factors $\tilde{\lambda}_{ij}$, in such a way as

$$\tilde{\lambda}_{12} \oplus \tilde{\alpha}_{12}(t) = \tilde{\lambda}_{1j} \oplus \tilde{\alpha}_{1j}(t) \qquad \qquad \text{for } j = 3, 4, \dots N$$

$$\{\tilde{\lambda}_{12} \oplus \tilde{\alpha}_{12}(t)\}^{\tilde{k}} = \{\tilde{\lambda}_{1j} \oplus \tilde{\alpha}_{1j}(t)\}^{\tilde{k}}$$
 for $k = 1, 2, ..., N-1$ (6')

where the "little circle" represents the first order *incipient* derivative. Such a specific notation was evidently chosen (and consequently adopted) in analogy to classical Newton's "dot" notation, usually used to indicate a first-order traditional derivative.

$$\left\{ \tilde{r} \right\}_{s} = e^{ \begin{bmatrix} \tilde{\alpha}_{13}(t) & \dots & \tilde{\alpha}_{1N}(t) \\ \tilde{\alpha}_{21}(t) & 0 & \dots & \tilde{\alpha}_{2N}(t) \\ \dots & \dots & \dots & \dots \\ \tilde{\alpha}_{N1}(t) & \tilde{\alpha}_{N2}(t) & \dots & 0 \end{bmatrix}}$$
(7).

Such a formal structure also allows us to assert that it is possible to choose, as a preferential reference perspective, any couple of elements of the Ordinal System, in order to give an equivalent representation of the same. Such a preferential choice introduces a further simplification, due to the fact that any preferential description adopted is "perfectly specular" to any other perspective specifically associated to each one of the remaining N-1 elements of the System. This evidently means that the description reduces to (N-1)(N-2)/2 distinct elements, which are coupled together in the form of "binary-duet" structures.

Under particular conditions, however, all these distinct basic elements can also be so strictly related to each other (in Ordinal terms) that the description can equivalently be given by means of one sole element (assumed as a preferential reference perspective) and only (N-1) correlating factors $\tilde{\lambda}_{ij}$.

Clearly, all these properties are exclusively related to the concept of Ordinal Matrix. These intrinsic properties, in fact, express a much more profound concept of "symmetry" (with respect to the traditional one), which, as already anticipated, can more appropriately be termed as "specularity". That very aspect which offers such relevant advantages when developing a computer code based on an Ordinal Model.

Appendix: mathematical procedure which leads to explicit solution (4)

The simplest (and most significant) way of presenting the formal procedure according to which Solution (4) can easily be obtained is that of adopting the following "convention": the three variables $\rho(t)$, $\varphi(t)$, $\varphi(t)$, $\varphi(t)$, will initially be represented without either the symbol of Ordinality (the "tilde" notation) or the "cardinal" time dependence "(t)". Consequently, they will simply be represented as ρ, φ, θ . In this way it will be possible to show how these variables progressively "emerge" from the System of Eqs. (1), (2), (3'), with their pertinent Ordinality and associated time-dependent cardinality.

This approach is thus mainly finalized to show that the Solution to the System made up of the three Fundamental Equations (1), (2), (3') is not a "cogent" and "necessary" consequence of the same, but is an "emerging" Solution. That is a Solution pertaining to a Generative Process, and thus characterized by a progressively Ascendant Ordinality and increasing Harmony.

The explicit Solution (4) is then obtained through the following subsequent passages:

i) by introducing the definition of the fundamental Relation Space (3') in the First and Second Basic Equations (1) and (2), the first one, after some simplifications, becomes

$$(\frac{\tilde{d}}{\tilde{d}t})^{(\tilde{2}\tilde{\ell}\tilde{2})^{2}}\rho^{2} \oplus 4\mathbb{R}(\mathring{\varphi}\mathbb{R}\tilde{j} \oplus \mathring{\theta}\mathbb{R}\tilde{k})\mathbb{R}(\frac{\tilde{d}}{\tilde{d}t})^{(\tilde{2}\tilde{\ell}\tilde{2})}\rho^{2} \oplus 4[\Theta(\mathring{\varphi})^{2}\Theta(\mathring{\theta})^{2} \oplus 2\mathring{\varphi}\mathbb{R}\mathring{\theta}\mathbb{R}\tilde{k}] = 0$$
(A1),

whereas Eq. (2) becomes

$$(\frac{\tilde{d}}{\tilde{d}t})^{(\tilde{2}/\tilde{2})^{\tilde{2}}} \varphi \otimes \tilde{k} \Theta(\frac{\tilde{d}}{\tilde{d}t})^{(\tilde{2}/\tilde{2})^{\tilde{2}}} \theta \otimes \tilde{j}) \oplus \frac{1}{\rho^{2}} \otimes (\frac{\tilde{d}}{\tilde{d}t})^{(\tilde{2})/(\tilde{2})} \rho^{2} \otimes [(\frac{\tilde{d}}{\tilde{d}t})^{(\tilde{2}/\tilde{2})} \varphi \otimes \tilde{k} \Theta(\frac{\tilde{d}}{\tilde{d}t})^{(\tilde{2}/\tilde{2})} \theta \otimes \tilde{j})] = 0 \quad (A2).$$

In both equations the symbol " Θ " represents the *specular concept* of the Ordinal "composition" (as a Whole) previously indicated by " \oplus " in Eqs. (3'), (6), (6').

For the sake of clarity such equations are initially written in the simple form $\{\{2/2\} \uparrow \{2 \uparrow\}\}$. This is because the Ordinality which appears in Eq. (2) allows us to consider the Over-Ordinality $\uparrow \tilde{N}$ in a second phase of the solution process;

ii) Equation (A1) is evidently obtained by considering the specific properties of the spinors previously mentioned. These properties are simply listed here below because they precisely represent the explicit consequence of their being spinors:

$$\tilde{i} \circledast \tilde{i} = \oplus 1$$
 $\tilde{i} \circledast \tilde{j} = \tilde{j}$ $\tilde{i} \circledast \tilde{k} = \tilde{k}$ (A3.1)

$$\tilde{j} \otimes \tilde{i} = \tilde{j}$$
 $\tilde{j} \otimes \tilde{j} = \Theta 1$ $\tilde{j} \otimes \tilde{k} = \tilde{k}$ (A3.2)

$$\tilde{k} \otimes \tilde{i} = \tilde{k}$$
 $\tilde{k} \otimes \tilde{j} = \tilde{k}$ $\tilde{k} \otimes \tilde{k} = \Theta 1$ (A3.3).

Such properties are also particularly apt to illustrate the afore-mentioned concept according to which \tilde{i} , \tilde{j} , \tilde{k} are very similar (albeit not strictly equivalent) to a system of three complex numbers, characterized by one real unit $\tilde{(i)}$ and two imaginary units $\tilde{(j)}$ and \tilde{k} (Giannantoni, 2007, ch. 6);

- iii) Equation (A2), in turn, is obtained by considering that the "vector" product " \otimes " is now referred to the three "spinors" previously defined, and thus characterized by the properties given by Eqs. (A3.1), (A3.2), (A3.3);
- iv) The solution to Eq. (A1) can then be obtained by considering ρ^2 as a function of φ and ϑ . In this way, the associated characteristic equation gives origin to a solution of Ordinality $(2/2)^2$, that is (Giannantoni, 2008a, ch. 22)

$$\left[\begin{pmatrix} \alpha_1 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \alpha_4 \end{pmatrix}\right]^2 \tag{A.4}$$

where
$$\alpha_1 = (A \oplus \gamma_1)^{(\tilde{2}/\tilde{2})}$$
 $\alpha_2 = (A \oplus \gamma_2)^{(\tilde{2}/\tilde{2})}$ for $\Delta = \varphi \otimes \theta > 0$ (A5.1)

and
$$\alpha_3 = (A \oplus \gamma_3)^{(\tilde{2}/\tilde{2})}$$
 $\alpha_4 = (A \oplus \gamma_4)^{(\tilde{2}/\tilde{2})}$ for $\Delta = \mathring{\phi} \otimes \overset{\circ}{\theta} < 0$ (A5.2),

in which:
$$\gamma_1 = \oplus 2\sqrt{\Delta} \, \mathbb{R} \, (\oplus 1\Theta \, \tilde{k})$$
 $\gamma_2 = \oplus 2\sqrt{\Delta} \, \mathbb{R} \, (\Theta 1 \oplus \tilde{k})$ (A5.3)

$$\gamma_3 = \bigoplus 2\sqrt{|\Delta|} \otimes (\bigoplus 1 \oplus \tilde{k}) \qquad \qquad \gamma_4 = \bigoplus 2\sqrt{|\Delta|} \otimes (\Theta 1 \oplus \tilde{k})$$
 (A5.4).

Consequently, the general solution to Eq. (A1) can synthetically be structured as follows, in terms of φ and θ (and the initial value ρ_0^2)

$$\tilde{\rho}^2 = \rho_0^2 \otimes e^{\int_0^1 \left[\left(\frac{\alpha_1}{\alpha_3} \right) \cdot \left(\frac{\alpha_2}{\alpha_4} \right) \right]^2 dt}$$
(A.6).

where the variable ρ^2 bigins to "emerge" with a preliminary form of its own Ordinality. This is because the latter is that same Ordinality which is still pertaining to the present state of a progressively "emerging" solution;

v) Afterwards, by adopting the following assignation relationship

$$\stackrel{\circ}{\theta} = \chi \stackrel{\circ}{\cdot} \varphi \tag{A.7}$$

obtained through the introduction of the correlating factor χ (whose Ordinality and cardinality will emerge later on from the same solution process), Eq. (A.6) becomes

$$\tilde{\rho}^2 = \rho_0^2 \mathbb{R} e^{\int_0^t \left[\begin{pmatrix} \alpha_1 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \alpha_4 \end{pmatrix} \right]^2 dt} = \rho_0^2 \mathbb{R} e^{\int_0^t M(\alpha_{ij})^2 dt}$$
(A.8).

In fact, as a consequence of condition (A.7), the Ordinal Matrix $M(\alpha_{ij})$ that appears in Eq. (A.8) will contain only those terms α_{ij} which depend on φ . The latter, in turn, is already thought of in terms of its future time-dependent cardinality. This is why the Ordinal Matrix $M(\alpha_{ij})$, for the sake of clarity, from now on will be renamed as $M_*(\alpha_{ij})$;

vi) By introducing such an expression into Eq. (A.2), and by taking into account assignation condition (A.7)), Eq. (A.2) becomes

$$(\frac{\tilde{d}}{\tilde{d}t})^{(\tilde{2}/\tilde{2})^{\tilde{2}}} \varphi \oplus (M_*) \otimes (\varphi)^{(\tilde{2}/\tilde{2})} \otimes (\frac{\tilde{d}}{\tilde{d}t})^{(\tilde{2}/\tilde{2})} \varphi = 0$$
(A.9),

which can also more explicitly be re-written as follows

$$(\frac{\tilde{d}}{\tilde{d}t})^{(\tilde{2}/\tilde{2})}(\frac{\tilde{d}}{\tilde{d}t})^{(\tilde{2}/\tilde{2})}\varphi \oplus (M_*) \otimes (\frac{\tilde{d}}{\tilde{d}t})^{(\tilde{2}/\tilde{2})}\varphi \otimes (\frac{\tilde{d}}{\tilde{d}t})^{(\tilde{2}/\tilde{2})}\varphi = 0$$
(A.10)).

This is exactly that equation from which the variable φ will "emerge" as $\varphi(t)$, that is with its proper *Ordinality* and its explicit associated *time-dependent cardinality*.

Equation (A.10) in fact can easily be solved by setting

$$\left(\frac{\tilde{d}}{\tilde{d}t}\right)^{(\tilde{2}/\tilde{2})}\varphi = \Phi \tag{A.11}.$$

In such a way Eq. (A.10) transforms into the following simpler equation

$$\left(\frac{\tilde{d}}{\tilde{d}t}\right)^{(\tilde{2}/\tilde{2})} \Phi \oplus (M_*) \otimes \Phi^{\tilde{2}} = 0 \tag{A.12}$$

which has the same form as Riccati's Equation, although written in terms of "incipient derivatives" and Ordinal terms in the unknown variable Φ .

Riccati's Equation, written in such a form, always has an explicit solution (Giannantoni, 2007, ch. 2). This is because traditional linear and non-linear differential equations, when reinterpreted in terms of incipient derivatives, always have explicit solutions (ib., ch. 3);

vii) Eq. (A.12) gives origin to the explicit solution $\Phi(t)$. This can be seen as an "emerging" solution, because it is now characterized by its proper *Ordinality* and explicit associated time-dependent cardinality.

As a consequence, the corresponding "emerging" solution to Eq. (A.11) can easily be obtained by replacing the auxiliary variable Φ by means of the obtained "emerging" solution $\Phi(t)$;

- viii) Analogously, by replacing φ into Eqs. (A.7) and (A.8) with the "emerging" solution $\varphi(t)$, we can immediately obtain the corresponding "emerging" solutions $\tilde{\vartheta}(t)$ and $\tilde{\rho}(t)$, respectively. This is because the Harmony Conditions pertaining to the considered sub-system of Ordinality $\{\tilde{2}/\tilde{2}\} \uparrow \{\tilde{2} \uparrow\}$ also contribute to the "emerging" of the explicit form of the "correlating factor" $\tilde{\chi}(t)$, with its pertaining Ordinality and time-dependent cardinality;
- ix) At this point the explicit time evolution of the proper space (3') is perfectly known, obviously when the latter is considered as being preliminarily referred to a sub-system of Ordinality $\{\tilde{2}/\tilde{2}\} \uparrow \{\tilde{2}\uparrow\}$;

x) The "emerging" solution to the System described by three Generative Equations (1), (2), (3'), understood as one sole entity, can then be obtained by considering the Over-Ordinality $\uparrow N$ pertaining to the Whole N-body System.

Such a solution, when appropriately restructured in an exponential form, precisely assumes the structure of Eq.

(4), in which any element α_{ij} is characterized by the Ordinality $\{\tilde{2}/\tilde{2}\} \uparrow \{\tilde{2}\uparrow\}$.

References

Giannantoni C., 2001a. The Problem of the Initial Conditions and Their Physical Meaning in Linear Differential Equations of Fractional Order. Applied Mathematics and Computation 141 (2003) 87-102.

Giannantoni C., 2001b. Mathematical Formulation of the Maximum Em-Power Principle. Second Biennial International Emergy Conference. Gainesville, Florida, USA, September 20-22, 2001, pp. 15-33.

Giannantoni C., 2002. The Maximum Em-Power Principle as the basis for Thermodynamics of Quality. Ed. S.G.E., Padua, ISBN 88-86281-76-5.

Giannantoni C., 2004. Mathematics for Generative Processes: Living and Non-Living Systems. 11th International Congress on Computational and Applied Mathematics, Leuven, July 26-30, 2004. Applied Mathematics and Computation 189 (2006) 324-340.

Giannantoni C., 2006. Emergy Analysis as the First Ordinal Theory of Complex Systems. Proceedings of the Fourth Emergy Conference 2006. Gainesville, Florida, USA, January 17-22, pp. 15.1-15.14.

Giannantoni C., 2007. Armonia delle Scienze (vol. I). La Leggerezza della Qualità. Ed. Sigraf, Pescara, Italy, ISBN 978-88-95566-00-9.

Giannantoni C., 2008a. Armonia delle Scienze (volume secondo). L'Ascendenza della Qualità. Ed. Sigraf, Pescara (Italy), ISBN 978-88-95566-18-4.

Giannantoni C., 2008b. From Transformity to Ordinality, or better: from Generative Transformity to Ordinal Generativity. Proceedings of the 5th Emergy Conference. Gainesville, Florida, USA, January 31-February 2, 2008.

Giannantoni C., 2009. Ordinal Benefits vs Economic Benefits as a Reference Guide for Policy Decision Making. The Case of Hydrogen Technologies. Energy n. 34 (2009), pp.2230–2239.

Giannantoni C., 2010a. The Maximum Ordinality Principle. A Harmonious Dissonance. Proceedings of the 6th Emergy Conference. Gainesville, USA, January 14-16, 2010.

Giannantoni C., 2010b. Protein Folding, Molecular Docking, Drug Design. The Role of the Derivative "Drift" in Complex Systems Dynamics. Proceedings of the 3rd International Conference on Bioinformatics, Valencia, Spain, January 20-24, 2010.

Giannantoni C., 2011. Bio-Informatics in the Light of the Maximum Ordinality Principle. The Case of Duchenne Muscular Dystrophy. Proceedings of 4th International Conference on Bioinformatics. Rome, January 26-29, 2011.