The "Incipient" Derivative

The introduction of a new concept of "derivative" is substantially due to the fact that the traditional derivative (*d/dt*) is nothing but the formal reflex of three fundamental pre-assumptions when describing physical-biological-social systems: i) *efficient causality*; ii) *necessary logic*; iii) *functional relationships*. Such an aprioristic perspective thus excludes, from its basic foundation, the possibility that any process output might ever show anything "extra", with respect to its corresponding input, as a consequence of the intrinsic (supposedly) *necessary*, *efficient* and *functional* dynamics of the system analyzed. Consequently, such a theoretical approach will never see any "output excess", exactly because it has already excluded from the very beginning (but only aprioristically) that there might be "any". In this sense it is possible to say that such an approach describes all the phenomena as they were mere "mechanisms".

On the contrary, Co-production, Inter-action and Feed-back Processes, that is the basic Processes which characterize self-organizing System, suggest we think of a different form of "causality", precisely because their outputs always show something in "excess" with respect to their inputs. This "causality" may be termed as "generative" causality or "spring" causality or whatsoever term is appropriate. In any case the basic concept is rather clear. Any term adopted is simply focused on indicating that it is worth supposing a form of "causality" which is capable of giving rise to something "extra" with respect to what it is usually foreseen (and expected) by the traditional approach.

The same happens for Logic. In fact, a different Logic is correspondently needed in order to contemplate the possibility of coming to conclusions much richer than their corresponding premises. This new form of Logic, in turn, could correspondently be termed as "adherent" Logic, because its conclusions always faithfully conform to the premises. Nonetheless, the conclusions could even be well-beyond what is strictly foreseen by the same premises when interpreted in strictly necessary terms.

As an adherent consequence of both previous concepts, the relationships between phenomena cannot be reduced to mere "functional" relationships between the corresponding cardinal quantities. This is because they always "vehicle" something else, which leads us to term those relationships as "Ordinal" relationships. The term "Ordinal" would thus explicitly remind us that each part of the System is related to the others exclusively because, above all, it is related to the Whole or, even better, it is "ordered" to the Whole.

Consequently, the new concept of derivative is nothing but the adherent "translation", in formal terms, of the three new gnosiological concepts: *Generative Causality*, *Adherent Logic* and *Ordinal Relationships*.

Such a new derivative was intentionally termed as "incipient" precisely because it describes the processes in their generating activity or, preferably, because it focuses on their pertinent outputs in their specific act of being born. Its mathematical definition (already presented in Giannantoni 2001a, 2002, 2004b, 2008a, 2009, 2010a, 2010b, 2011a, 2011b). is here recalled only for the sake of clarity

$$\frac{\tilde{d}^q}{\tilde{d}t^q} f(t) = \underset{\tilde{\Delta}t:0\to 0^+}{\tilde{Lim}} \circ \left(\frac{\tilde{\delta}-1}{\tilde{\Delta}t} \right)^{\tilde{q}} \circ f(t) \qquad \textbf{(1).}$$

Its structure appears as being substantially "similar" to the traditional derivative, even if it is deeply different. The adoption of the "tilde" notation indicates that the same symbols are now understood in a substantially different way. To this purpose, before illustrating the proper meaning of definition (1), it is worth noting that the traditional increment $\Delta f(t) = f(t+\Delta t) - f(t)$ can equivalently be expressed in terms of the *operator* δ, which represents the variation $\delta f(t) = f(t+\Delta t)$ of the analyzed property $f(t)$:

$$\Delta f(t) = f(t+\Delta t) - f(t) = (\delta - 1) f(t) \qquad \textbf{(2).}$$

Thus the ratio $\frac{\tilde{\delta}-1}{\tilde{\Delta}t}$ (3) (in definition (1)) substantially replaces the traditional incremental $\frac{\Delta}{\Delta t}$. The symbol "tilde", however, should remind us that its meaning is now completely different.

The comparison between the "incipient" derivative and the traditional derivative can better be illustrated by first pointing out that the latter corresponds to an "operative" definition, because the priority of the operators that constitute its definition is understood from right to left, that is: i) the concept of function (which is assumed to be a primary concept); ii) the incremental ratio (of the supposedly known function); iii) the operation of limit (referred to the result of the previous two steps).

The "incipient" derivative, on the contrary, is based on the *direct priority* of the order of the three elements that constitute its definition (from left to right). This is why they acquire a completely different meaning. Let us start from the symbol $\tilde{Lim}$. The etymological origin of the word can help us: "Limit" comes from the Latin word "Limen", which means a "threshold". It could be a "threshold" of a door or of a "window", from which we

observe and describe the considered phenomenon. In such a case the symbol $\tilde{\Delta} t : 0 \to 0^+$ indicates not only the initial time of our registration, but also the proper "origin" (in its etymological sense) of something new which we observe (and are going to describe) in its proper act of being born. It becomes then evident that the "operator" $\tilde{\delta}$ now registers the variation of the observed $f(t)$, not only in terms of quantity, but also, and especially, in terms of Quality (as the symbol "tilde" expressly reminds us). Thus the ratio (3) indicates not only a quantitative variation in time, but both the variation in Quality and quantity. In fact, from the very beginning of any process we recognize its specific genesis in the form of a Co-production, Inter-action and Feed-back, respectively. We can then take explicit note of this genetic property by means of a rational number as an exponent of the Ordinal Incremental Ratio: 1/2, 2, and {2/2} respectively. Consequently, when we take the incipient (or "prior") derivative of any $f(t)$, this will keep the "memory" of its genetic origin because, besides its quantity, it will result as being structured according the indication of such an exponent. This is correspondently termed as Ordinality, because it precisely expresses (as already anticipated) how each part of the output is related to all the others or, better, how it is genetically Ordered within the context of the Whole. In this way the corresponding output "functions" ("binary", "duet", and "binary-duet" functions, respectively) are structured in such a way as to show that "excess" of Information which cannot be accounted for by means of traditional derivatives, because it is never reducible to its sole phenomenological premises or to our traditional mental categories (Giannantoni 2004a, 2008a, 2009, 2010a). In other terms, the "incipient" derivative represents the Generativity of the considered Process, that is the output "excess" (per unit time) characterized by both its Ordinality and its related cardinality. This is also the reason why the sequence of the symbols (in Eq. (1)) is interpreted as a generative inter-action (see the symbol "$\circ$") between the three considered concepts. In this way the "incipient" derivative is also able to unify (and, at the same time, to specify) the three basic Processes (previously recalled), now explicitly understood in terms of Quality.

The Generativity concept, in fact, is that which unifies the three Processes, whereas the pertinent Ordinality expresses the structure of the corresponding output "functions" (as "binary", "duet", and "binary-duet" functions, respectively), which are understood as a Whole (ib.).

The adoption of "incipient" derivatives, however, is not exclusively restricted to the three afore-mentioned Processes, because definition (1) is valid for *any* fractional number q. This suggests we may also adopt such a definition to model *any* complex System, by simply considering "incipient" derivatives characterized by those rational numbers (m/n) which result as being more appropriate to each specific System analyzed.

Mathematical Appendix

This Appendix is devoted to show how it is possible to generalize, under dynamic conditions, the three basic Generative Processes pointed out by H.T. Odum (1994a, b, c), generally modeled in terms of Emergy Algebra (Brown & Herendeen, 1996). To this aim, and for the sake of simplicity, we can always refer to Ordinal relationships represented by exponential functions (in the most general form $e^{\alpha(t)}$), because, as is well known, any function $f(t)$ can always be written as $f(t) = e^{\ln f(t)}$ (A.1) and, consequently, in the exponential form $f(t) = e^{\alpha(t)}$ (A.2), where $\alpha(t) = \ln f(t)$ (A.3).

A) Co-Production Process

This Process, schematically graphed in Fig. A.1, can formally be represented by means of a derivative of order 1/2. This derivative, in fact, gives rise to a "binary" function, that is: an output made up of two distinct entities, which however form *one sole thing*. This is equivalent to say that the two "by-products", precisely because generated by the same *unique* (Generative) Process, keep memory of their common and *indivisible origin*, even if they may have, later on, completely different topological locations in time.

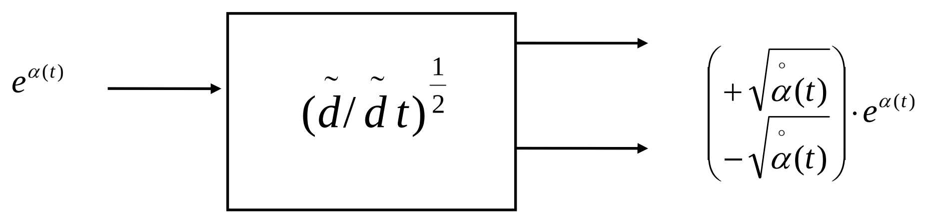

Fig. A.1 – Representation of a Co-Production Process

where $\overset{\circ}{\alpha}(t)$ represents the first order *incipient* derivative of the function $\alpha(t)$ (see also later on).

The genesis of "binary" functions (from a Co-production Process) can formally be represented as:

$$(\frac{\tilde{d}}{\tilde{d}\,t})^{\frac{1}{2}} e^{\alpha(t)} = \begin{pmatrix} +\sqrt{\mathring{\alpha}(t)} \\ -\sqrt{\mathring{\alpha}(t)} \end{pmatrix} \cdot e^{\alpha(t)} \qquad \textbf{(A.4),}$$

where the order of the derivative 1/2 explicitly reminds us that the output generated is "1" sole entity, although made up of "2" parts. In other terms the output, when understood as a whole, is much more than the simple sum of its single parts. Said differently, the *uniqueness* of the Generative Process, recognized as being a specific property of a Co-generation Process, remains as being *in-divisible*, and thus also *ir-reducible* to the component parts.

A simple example of such a Generative Process can be represented by the Generation of two "twins", who always keep "trace" of their common Co-generation, not only at a genetic level, but also through several other characteristics.

Such an example can also be useful to illustrate that the corresponding equivalent of the above-mentioned genetic properties, can be represented, at a formal level, by the square root $\sqrt{\mathring{\alpha}(t)}$. This in fact represents a sort of "extraction" (on behalf of the derivative of order 1/2) of the "genetic properties" of the given Ordinal relationship $e^{\alpha(t)}$, whereas the symbols "+/-" characterize the corresponding distinct cardinalities (in reality, at a more general level of representation, these symbols will lose their algebraic sense, to assume a deeper meaning of internal relationships, and thus represented differently, for instance, as "$\tilde{\oplus}/\tilde{\Theta}$" (Giannantoni, 2008b)).

The concept of Co-generation Process, however, is not limited to living beings. This Generative Process, in fact, is also present in Classical Mechanics. Such a model, in fact, when adopted to describe the relationship between Sun and Mercury, understood as being generated by the same Laplace Nebula, is able to explain the famous Mercury's Precessions, by always keeping the same structure of Newtonian Laws, without any necessity of adopting General Relativity (ib.) The same happens in Quantum Mechanics, where the same Co-generative model is able to interpret the famous (and still unexplained) "Entanglement" of two photons *co-generated* by the same process (ib.).

What's more it is also ever-present in Social Sciences and, in particular, in Economics, precisely when the same Productive activity generates two or more "by-products".

B) Inter-Action Process

This Generative Process can easily be illustrated by considering first a single input Process (see Fig. A.2). In such a case the Process, modeled through the incipient derivative of Order 2, represents a *reinforcement* of the same input, so giving rise to a new entity which, however, is much more than the simple (cardinal) product of the original input by itself considered, and it can be thus represented as

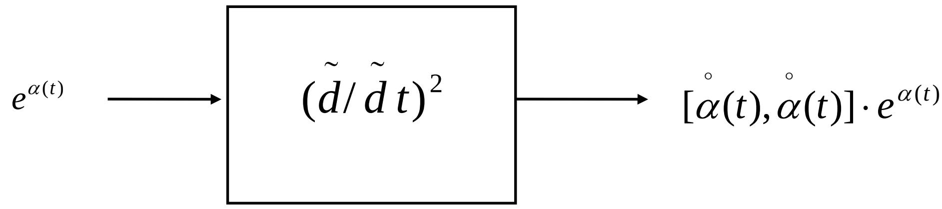

Fig. A.2 – Formal representation of a "duet" Process Amplification

This Process can be termed as "Generative" precisely because the two contributions not only reinforce each other, but are also unified in a new *one sole entity*. In other terms, they not only increase the cardinality of their joint action, but also generate an exceeding Quality, represented by the *uniqueness* and *irreducibility* of their co-operating activity, because *solidly* and *indissolubly* orientated in the same "direction". This is why the corresponding output can be termed as a "duet" function and represented, in formal terms, as follows

$$(\frac{\tilde{d}}{\tilde{d}\,t})^{2} e^{\alpha(t)} = [\mathring{\alpha}(t), \mathring{\alpha}(t)] \cdot e^{\alpha(t)} \qquad \textbf{(A.5).}$$

It is then easy to recognize that, only when such a Process is seen in mere *cardinal* terms, does the output reduce to the traditional result of a *scalar* product between the two quantities $\mathring{\alpha}(t)$, by giving

$$(\frac{d}{dt})^2 e^{\alpha(t)} = [\mathring{\alpha}(t)\cdot\mathring{\alpha}(t)]\cdot e^{\alpha(t)} = [\mathring{\alpha}(t)]^2 \cdot e^{\alpha(t)} \quad \textbf{(A.6).}$$

In such a case, in fact, the process is coherently described by means of the traditional derivative (see Eq. (A.6)), which, as repeatedly asserted, "filters" any form of Ordinality.

B1) The Inter-Action Process in its proper sense

The Inter-Action Process, in its proper definition, manifests its true essence in the presence of (at least) two distinct inputs and it can be thus represented as in Fig. A.3.

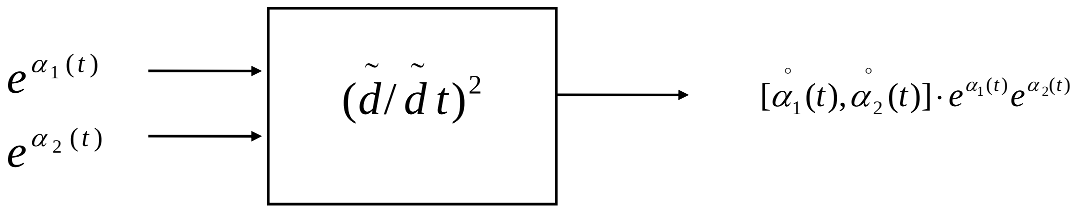

Fig. A.3 - Representation of an Inter-Action Process

where the "duet" $[\mathring{\alpha}_1(t), \mathring{\alpha}_2(t)]$ **now stands for the Logic "and":** $[\mathring{\alpha}_1(t), \mathring{\alpha}_2(t)] \wedge [\mathring{\alpha}_2(t), \mathring{\alpha}_1(t)]$.

It is thus characterized by the total absence of any form of internal reciprocal priority.

It also worth mentioning that the Inter-Action Process is very frequently associated to a Co-generation Process. In such a case we can also speak of an Inter-Action Process characterized by a "subjacent" Co-generation Process (with its associated "binary" function). The Process can be then characterized by a derivative of Order 2/2 and thus represented as in Fig. A.4

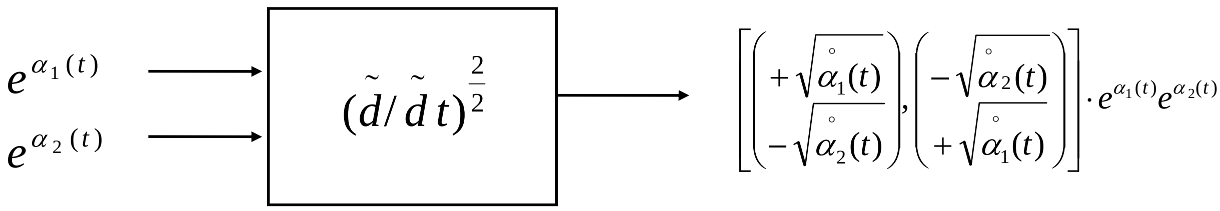

Fig. A.4 - Representation of an Inter-Action Process (with a "subjacent" Co-generation)

In such a case the two inputs not only contribute to a reciprocal reinforcement, but are also reciprocally coupled in the form of a "binary" function. In addition, such a coupling, is further enhanced by the inter-exchange (and successive coupling) of the specific "genetic" properties of the input Ordinal functions (see $\sqrt{\mathring{\alpha}_1(t)}$ **and** $\sqrt{\mathring{\alpha}_1(t)}$ **, respectively). The Process thus gives rise to a "duet-binary" function:**

$$(\frac{\tilde{d}}{\tilde{d}\,t})^{\frac{2}{2}}[e^{\alpha_1(t)}e^{\alpha_2(t)}] = \left[\begin{pmatrix} +\sqrt{\mathring{\alpha}_1(t)} \\ -\sqrt{\mathring{\alpha}_2(t)} \end{pmatrix}, \begin{pmatrix} -\sqrt{\mathring{\alpha}_2(t)} \\ +\sqrt{\mathring{\alpha}_1(t)} \end{pmatrix}\right]\cdot e^{\alpha_1(t)}e^{\alpha_2(t)} \quad \textbf{(A.7).}$$

A significant example of this Generative Process can be represented by the generation of a living being. The formal expression (A.7), in fact, would be a preliminary representation of the re-composition of a *completely new* couple of chromosomes by starting from one chromosome pertaining to the father and the other pertaining the mother. Evidently, the Process is here extremely simplified. In fact, in the human case (for instance) we should have to consider 23 couples of chromosomes deriving from the father and 23 from the mother, respectively, which give rise to a *completely new* human being, characterize by 46 new couples of chromosomes.

C) Ordinal Feed-Back

This Process can easily be illustrated on the basis of the Inter-Action Process, by assuming that the Ordinal output of the Process contributes, together with the input, to its same genesis (see Fig. A.5)

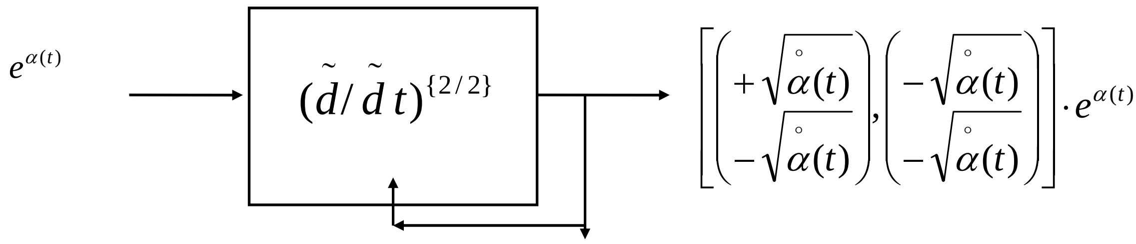

Fig. A.5 - Representation of an Ordinal Feed-Back Process

In such a case the output represents a *perfect specular* reproduction of the input, although at a *higher* Ordinality level. This is why the derivative of Order {2/2} is specifically represented in brackets: to expressly point out such a specific *harmonic consonance* between the input and the output of the Ordinal Feed-back Process, which can be represented in formal terms as follows

$$(\frac{\tilde{d}}{\tilde{d}\,t})^{\{2/2\}} e^{\alpha(t)} = \left[\begin{pmatrix} +\sqrt{\mathring{\alpha}(t)} \\ -\sqrt{\mathring{\alpha}(t)} \end{pmatrix}, \begin{pmatrix} -\sqrt{\mathring{\alpha}(t)} \\ -\sqrt{\mathring{\alpha}(t)} \end{pmatrix}\right] \cdot e^{\alpha(t)} \qquad \textbf{(A.8).}$$

At this stage, Eqs. (A.4), (A.5) and (A.8) represent the formal generalization of the Rules of Emergy Algebra (Brown & Herendeen, 1996) corresponding to the three mentioned Generative Processes. They also show, in each case, the pertinent genesis of an *excess of Quality*. In fact, Co-production Transformity is now replaced by the *Ordinality* 1/2 (that is the power of the derivative ($\tilde{d}/\tilde{d}\,t$) understood in an Ordinal sense). The same happens for the Inter-action Process, now represented by the incipient derivative of order 2. Finally, the most elementary Feed-back Process is represented by the incipient derivative of order {2/2}, understood as a unique formal entity.

In this respect it is worth noting that such an Ordinality {2/2} does not correspond to the cardinal value of "1", nor does it correspond to 2/2, because the Ordinal Feed-back Process is not reducible to a simple "combination" of the two previous Processes. The same Eq. (A.8) clearly expresses, by itself, such an "excess" of Ordinality with respect to Eq. (A.7). The former in fact represents an "excess" in the *interior harmony relationships* due to *the persistence of form* (see later on) which intimately relates to each other the four distinct elementary functions which appear on its right hand side, now organized in *one sole* irreducible structure.

This can be easily understood by the fact that, in the most general case, the *incipient* derivative of order m/n is given by

$$(\frac{\tilde{d}}{\tilde{d}\,t})^{m/n} e^{\alpha(t)} = [\mathring{\alpha}(t)]^{(m/n)} \cdot e^{\alpha(t)} \qquad \textbf{(A.9)}$$

where $\mathring{\alpha}(t)$, as already said, represents the first order *incipient* derivative of the function $\alpha(t)$ and $[\mathring{\alpha}(t)]^{(m/n)}$ represents a multiple binary-duet function of Ordinality (m/n).

The little circle characterizing the incipient derivative $\mathring{\alpha}(t)$ was evidently chosen in analogy to classical Newton's "dot" notation, usually adopted to indicate a first-order derivative.

The different symbology is here justified by the fact that the former should now remind us the *conceptual difference* between the incipient derivative and the traditional one. In fact, even if $\mathring{\alpha}(t)$ and $\dot{\alpha}(t)$ coincide from a pure cardinal point of view, they are, on the contrary, radically different from a *Generative* (and also Ordinal) point of view. The former, in fact, represents the specific exit of a *Generative* Process, whereas the latter is always understood as the result of a *necessary* process (thought of as being a "mechanism" or a set of "mechanisms").

Moreover, such a purely quantitative coincidence is strictly valid only for $n = 1$. In fact, in the general case of an Ordinal exponent (m/n), Eq. (A.9) shows all the significance of its output Ordinal structure (in terms of multiple binary-duet functions) and, at the same time, the deep difference with respect to the corresponding *cardinal* fractional derivative of order m/n usually considered in Literature (Oldham & Spanier, 1974).

In addition, the right hand side of Eq. (A.9) reveals an extremely important property: a sort of "persistence of form". This exactly because it represents an "adherent" consequence of a Generative Process, characterized by

specific generation modalities. In other words, any "generating process" (modeled by the left hand side of Eq. (A.9)) gives origin to an Ordinal output (characterized by the Ordinality (m/n)) which corresponds to a multiple structure functions (described by the right hand side of Eq. (A.9)). These functions are similar to harmonic evolutions always in "resonance" (as in a "musical chord") with the original function and at the same time with each other, and they reach their *maximum harmony* in the case of a perfect Ordinal Feed-back $\{n/n\}$.

Such resonance relationships (whose number and typology are defined by the Ordinality (m/n)), when formalized in explicit terms, represent the afore-mentioned *interior harmony relationships.* These in fact express particular "coupling conditions" between integer and fractional derivatives (Giannantoni, 2004b). For example

$$(\frac{\tilde{d}}{\tilde{d}t})^{(1/2)} f(t) \circ (\frac{\tilde{d}}{\tilde{d}t})^{(1/2)} f(t) = f(t) \circ (\frac{\tilde{d}}{\tilde{d}t})^{(2/2)} f(t) = (\frac{\tilde{d}}{\tilde{d}t})^{(2/2)} f(t) \circ f(t) \qquad \textbf{(A.10),}$$

which is always valid, also under *steady state* conditions

$$(\frac{\tilde{d}}{\tilde{d}t})^{(1/2)} f(0) \circ (\frac{\tilde{d}}{\tilde{d}t})^{(1/2)} f(0) = f(0) \circ (\frac{\tilde{d}}{\tilde{d}t})^{(2/2)} f(0) = (\frac{\tilde{d}}{\tilde{d}t})^{(2/2)} f(0) \circ f(0) \qquad \textbf{(A.11),}$$

and for any function $f(t)$. In fact, all the above-mentioned properties, previously illustrated with reference to the simple exponential function $e^{\alpha(t)}$, can be easily generalized to any given function $f(t)$ on the basis of Eq. (A.3).

Consequently, the concepts of Co-production, Inter-action and Feed-back, initially illustrated by means of Eqs. (A.4), (A.5), (A.8), can always be adopted to describe *any* dynamic Generative Process, however complex it is. This also due to the fact that, while the right hand sides of Eqs. (A.4), (A.5), (A.8) represent the Ordinal structure of Co-production, Inter-action and Feed-back Processes, respectively, the corresponding left hand sides have an *identical structure*, always in the form $(\tilde{d}/\tilde{d}t)^q$, where q is a rational number which assumes the values of 1/2, 2 and {2/2}, respectively. This means that all Generative Processes are characterized by the same "sub-jacent" Generativity, which, however, can assume different forms, according to the Ordinality *q*. That is, a Generativity of Ordinal nature, because characterized by a specific Ordinality since the very beginning of the Process. This enables us to assert that Generative Transformity (generally and properly defined under steady state conditions) is nothing but a reflex of an Ordinal Generativity.

In fact, it is worth pointing out that all such properties are also valid under "steady-state" conditions. This is due to the fact that any Process, even in such conditions, is always the exit of a Generative activity. Thus any "constant" value describing its "steady-state" conditions has always the same form as (A.3), that is

$$f(t) = const = e^{\ln const} = e^{\phi(t)} \qquad \textbf{(A.12)}$$

where $\phi(t)$ has to be adherently and properly thought of as

$$\phi(t) = (\ln const) \cdot \tilde{1}(t) = (\ln const) \cdot \tilde{\int_0^t} \tilde{\delta}(\tau) \cdot d\tau \qquad \textbf{(A.13),}$$

where $\tilde{1}(t)$ corresponds to the Heaviside function (for $t \geq 0^+$), and $\tilde{\delta}(t)$ is the incipient Dirac Delta function, which coincides with the traditional Delta function only for $t \geq 0^+$.

Such a more general modeling capacity of *incipient* derivatives, associated with the afore-mentioned property that *any* Ordinal dynamic model always presents an explicit solution in a *closed form* [5], confers to the Incipient Differential Calculus much wider potentialities with respect to the Traditional Differential Calculus [ib.]. This is also confirmed by the fact that such a new mathematical approach led us to the solution of the famous "Three-body Problem" (in Classical Mechanics) (Giannantoni, 2007a, 2008a), "Protein Folding" (in Biology and Pharmacology) (Giannantoni 2010b, 2011a), and the "Three-good Two Factor Problem" (in Neo-Classical Economics), whose solution was afterward generalized to the case of the "Three-good N Factor Problem" too (Giannantoni, 2011b).

“Incipient” Derivative and Traditional Fractional Derivative

The “incipient” derivative is profoundly different from the traditional derivative, both of integer and fractional order. The latter in fact is an attempt at extending (and possibly generalizing) the concept of integer derivative. Such an attempt, however, as a consequence of the “necessary” logic which is always subjacent to the traditional approach, leads to a definition that does not substantially introduce anything new (in its consequential deductions) with respect to the traditional derivative of integer order (apart from some advantages in particular circumstances).
In fact “*It should be stressed that, because fractional derivatives and integrals can always be expressed using ordinary derivatives and integrals, any result obtainable through the fractional calculus may also be derived making use only of the concepts and symbolism of classical calculus*.” (Oldham & Spanier, 1974, p. xii).
The introduction of the “incipient” derivative, on the contrary, represents an attempt to overcome such limitations. In fact, as already shown, the “incipient” derivative of order (1/2), for instance, is aimed at representing the Generative activity, under dynamic conditions, of a *one sole* Co-Production System (indicated by the symbol “1”), which represents, by itself, something “*more*” than the simple sum of their parts (represented by the symbol “2”). In other words, the “incipient” derivative is finalized at translating, in formal terms, the fact that: *there are processes, in Nature, which cannot be considered as being pure “mechanisms”*.

A significant way of showing such a profound difference is that of obtaining the two different definitions (both the “incipient” derivative and the traditional derivative) by starting from the same “origin” (e. g. Faà di Bruno’s Formula), when the latter, however, is considered according to two *completely different* perspectives.
As is well known the traditional derivative of order *n* of a function of a function $G[f(x)]$ can be expressed by means of Faà di Bruno’s formula (1859)

$$D_x^n G[f(x)] = \sum \frac{n!}{k_1!k_2!\ldots k_n!} \cdot D_y^p G(y) \cdot \left(\frac{f'}{1!}\right)^{k_1} \cdot \left(\frac{f''}{2!}\right)^{k_2} \cdot\cdot \left(\frac{f^{(n)}}{n!}\right)^{k_n} \qquad \textbf{(A.14)}$$

where D_x represents the traditional derivative (d/dx) and the sum extends to all the partitions $(P_1, P_2, \ldots, P_n)$ of the integer m such as: $P_1 + P_2 + \ldots + P_n = m$ and $P_1 + 2P_2 + 3P_3 + \ldots + nP_n = n$ (Oldham & Spanier, 1974, p. 37).
If we now consider the reference exponential function (A.2), Eq. (A.14) gives

$$\frac{d^n}{dt^n} e^{\alpha(t)} = e^{\alpha(t)} \cdot \sum \frac{n!}{k_1!k_2!\ldots k_n!} \cdot \left(\frac{\dot{\alpha}}{1!}\right)^{k_1} \left(\frac{\ddot{\alpha}}{2!}\right)^{k_2} \cdots \left(\frac{\alpha^{(n)}}{n!}\right)^{k_n} \qquad \textbf{(A.15),}$$

which is nothing but the result of the well-known “chain rule”, because Eq. (A.15) is obtained on the basis of the well-known “step by step” derivation process.
If, on the contrary, the symbol D_x is understood as the “incipient” derivative and the function $e^{\alpha(t)}$ is understood as the cardinal reflex of an “emerging” Ordinal Relationship, Eq. (A.15) gives

$$\left(\frac{\tilde{d}}{\tilde{d}t}\right)^n e^{\alpha(t)} = e^{\alpha(t)} \cdot [\overset{\circ}{\alpha}(t)]^n \qquad \textbf{(A.16).}$$

In such a case, in fact, all the derivatives are *co-instantaneous*, and there are no partitions to be considered, because all the derivatives, as a consequence of their *persistence of form* (see Eq. (A.16)), are all harmoniously referred to the same Maximum Ordinality *n* (each time considered).

Moreover, as already anticipated, $\overset{\circ}{\alpha}(t)$ indicates that the second hand side of Eq. (A.16) is the exit of a *Generative* Process, whereas $\dot{\alpha}(t)$ is always understood as the result of a *necessary* process (thought of as being a “mechanism” or a set of “mechanisms”).
This is also the reason why, for a clearer distinction between the two different concepts of derivatives, the “fractional” number $q = (m/n)$, which appears in Eq. (3) and, consequently, in all the other equations, should better be represented with the tilde notation $\tilde{q} = (\tilde{m}, \tilde{m})$. Albeit the same symbol of “incipient” derivative $(\tilde{d}/\tilde{d}t)$ “qualifies”, by itself, any associated exponent as being, in turn, a symbol of Ordinality.

References

Brown M. T. and Herendeen R. A., 1996. Embodied Energy Analysis and Emergy analysis: a comparative view. Ecological Economics 19 (1996), 219-235.

Giannantoni C., 2001a. The Problem of the Initial Conditions and Their Physical Meaning in Linear Differential Equations of Fractional Order. Applied Math. and Computation 141 (2003) 87-102.

Giannantoni C., 2001b. Mathematical Formulation of the Maximum Em-Power Principle. 2nd Biennial International Emergy Conference. Gainesville, Florida, USA, September 20-22, 2001, pp. 15-33.

Giannantoni C., 2002. The Maximum Em-Power Principle as the basis for Thermodynamics of Quality. Ed. S.G.E., Padua, ISBN 88-86281-76-5.

Giannantoni C., 2004a. Differential Bases of Emergy Algebra. 3rd Emergy Evaluation and Research Conference, Gainesville, Florida, USA, January 29-31, 2004.

Giannantoni C., 2004b. Mathematics for Generative Processes: Living and Non-Living Systems. 11th International Congress on Computational and Applied Mathematics, Leuven, July 26-30, 2004. Applied Mathematics and Computation 189 (2006) 324-340.

Giannantoni C., 2006. Emergy Analysis as the First Ordinal Theory of Complex Systems. Proceedings of the Fourth Emergy Conference 2006. Gainesville, Florida, USA, January 17-22, pp. 15.1-15.14.

Giannantoni C., 2007a. Armonia delle Scienze (vol. I). La Leggerezza della Qualità. Ed. Sigraf, Pescara, Italy, ISBN 978-88-95566-00-9.

Giannantoni C., 2008a. From Transformity to Ordinality, or better: from Generative Transformity to Ordinal Generativity. Proceedings of the Fifth Emergy Conference. Gainesville, Florida, USA, January 31-February 2, 2008.

Giannantoni C., 2008b. Armonia delle Scienze (vol. II). L'Ascendenza della Qualità. Edizioni Sigraf, Pescara, Italy, ISBN 978-88-95566-18-4.

Giannantoni C. and Zoli M., 2009. The Derivative "Drift" in Complex Systems Dynamics. The Case of Climate Change and Global Warming. 22nd ECOS International Conference, Foz do Iguassu, Brazil, August 31-September 3, 2009.

Giannantoni C., 2009. Ordinal Benefits vs Economic Benefits as a Reference Guide for Policy Decision Making. The Case of Hydrogen Technologies. Energy n. 34 (2009), pp. 2230–2239.

Giannantoni C., 2010a. The Maximum Ordinality Principle. A Harmonious Dissonance. Proceedings of the 6th Emergy Conference. Gainesville, USA, January 14-16, 2010.

Giannantoni C., 2010b. Protein Folding, Molecular Docking, Drug Design. The Role of the Derivative "Drift" in Complex Systems Dynamics. Proceedings of the 3rd International Conference on Bioinformatics, Valencia, Spain, January 20-24, 2010.

Giannantoni C., 2011a. Bio-Informatics in the Light of the Maximum Ordinality Principle. The Case of Duchenne Muscular Dystrophy. Proceedings of 4th International Conference on Bioinformatics. Rome, January 26-29, 2011.

Giannantoni C., 2011b. Oeco-Nomics in the Light of the Maximum Ordinality Principle The N-Good and Three Factor Problem. Third International Workshop in Cleaner Production, Sao Paulo, Brazil, May 18-20, 2011.

Odum H. T., 1988. Self-Organization, Transformity and Information Science, v. 242, pp. 1132-1139, Nov. 25.

Odum H. T., 1994a. Ecological and General Systems. An Introduction to Systems Ecology. Re. Edition. University Press Colorado.

Odum H. T., 1994b. Environmental Accounting. Environ. Engineering Sciences. University of Florida.

Odum H. T., 1994c. Self Organization and Maximum Power. Environmental Engineering Sciences. University of Florida.

Oldham K. B. and Spanier J., 1974. The Fractional Calculus. Theory and Applications of Differentiation and Integration to Arbitrary Order. Academic Press, Inc., London.